HIROIMONO Is NP-Complete

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Abstract. In a Hiroimono puzzle, one must collect a set of stones from a square grid, moving along grid lines, picking up stones as one encounters them, and changing direction only when one picks up a stone. We show that deciding the solvability of such puzzles is **NP**-complete.

1 Introduction

Hiroimono (拾い物, "things picked up") is an ancient Japanese class of tour puzzles. In a Hiroimono puzzle, we are given a square grid with stones placed at some grid points, and our task is to move along the grid lines and collect all the stones, while respecting the following rules:

1. We may start at any stone.
2. When a stone is encountered, we must pick it up.
3. We may change direction only when we pick up a stone.
4. We must not make 180° turns.

Figure 1 shows some small example puzzles.

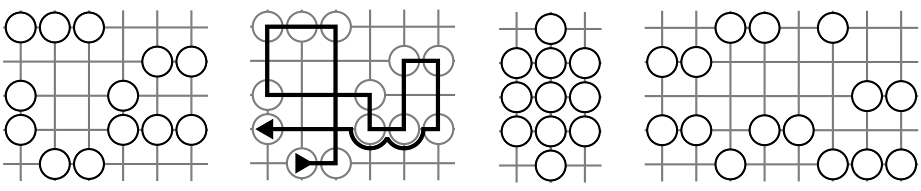

Fig. 1. (a) A Hiroimono puzzle. (b) A solution to (a). (c) Unsolvable. (d) Exercise.

Although it is more than half a millennium old [1], Hiroimono, also known as Goishi Hiroi (碁石ひろい), appears in magazines, newspapers, and the World Puzzle Championship. Many other popular games and puzzles have been studied from a complexity-theoretic point of view and proved to give rise to hard computational problems, e.g. Tetris [2], Minesweeper [3], Sokoban [4], and Sudoku (also known as Number Place) [5]. We shall see that this is also the case for Hiroimono.

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We will show that deciding the solvability of a given Hiroimono puzzle is **NP**-complete and that specifying a starting stone (a common variation) and/or allowing 180° turns (surprisingly uncommon) does not change this fact.

Definition 1. HIROIMONO *is the problem of deciding for a given nonempty list of distinct integer points representing a set of stones on the Cartesian grid, whether the corresponding Hiroimono puzzle is solvable under rules 1–4. The definition of* START-HIROIMONO *is the same, except that it replaces rule 1 with a rule stating that we must start at the first stone in the given list. Finally,* 180-HIROIMONO *and* 180-START-HIROIMONO *are derived from* HIROIMONO *and* START-HIROIMONO*, respectively, by lifting rule 4.*

Theorem 1. *All problems in Definition 1 are* **NP***-complete.*

These problems obviously belong to **NP**. To show their hardness, we will construct a reduction from 3-SAT [6] to all four of them.

2 Reduction

Suppose that we are given as input a CNF formula $\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ with variables $x_1, x_2, \ldots, x_n$ and with three literals in each clause. We output the puzzle p defined in Fig. 2–4. Figure 5 shows an example.

p :=

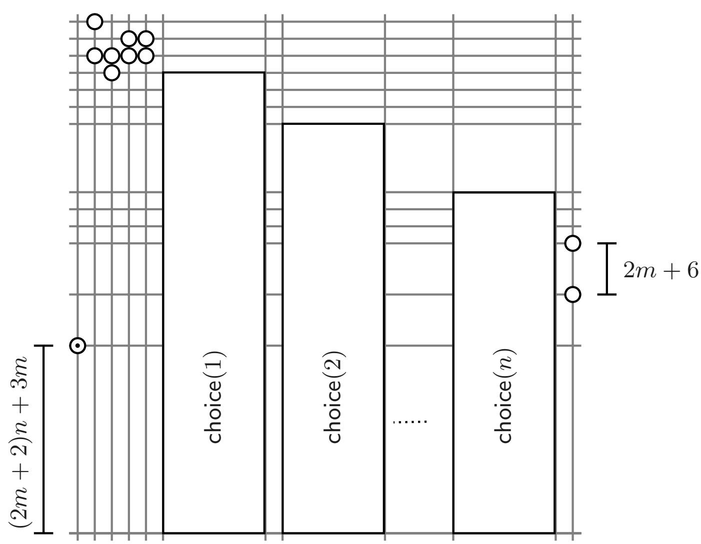

Fig. 2. The puzzle p corresponding to the formula ϕ. Although formally, the problem instances are ordered lists of integer points, we leave out irrelevant details such as orientation, absolute position, and ordering after the first stone ⊙.

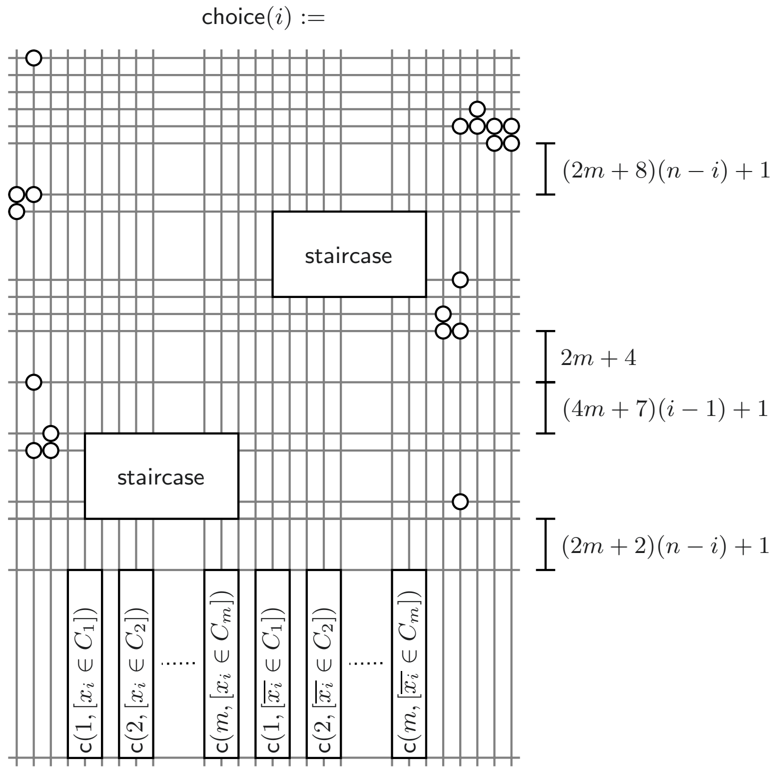

Fig. 3. The definition of choice(i), representing the variable x_i. The two staircase-components represent the possible truth values, and the c-components below them indicate the occurrence of the corresponding literals in each clause.

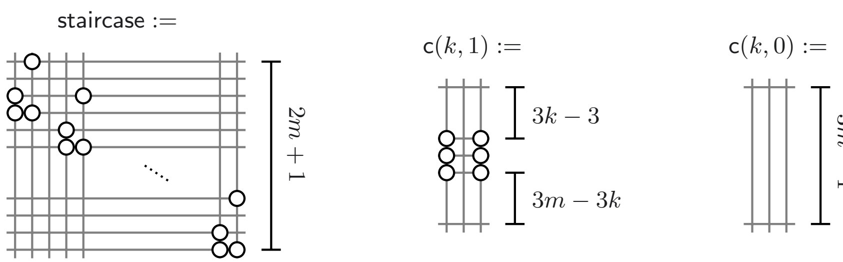

Fig. 4. The definition of staircase, consisting of m "steps", and the c-components. Note that for any fixed k, all c$(k,1)$-components in p, which together represent C_k, are horizontally aligned.

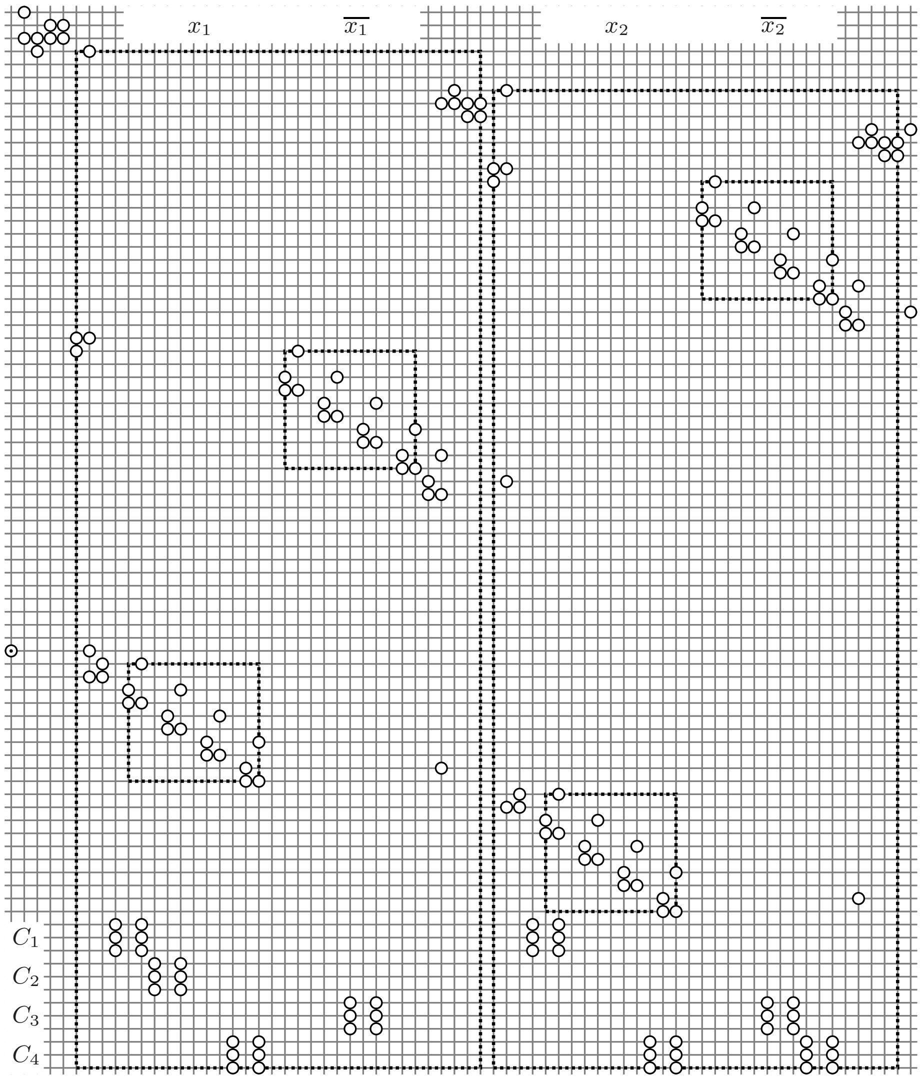

Fig. 5. If $\phi = (x_1 \vee x_2 \vee x_2) \wedge (x_1 \vee x_1 \vee x_1) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_2}) \wedge (x_1 \vee x_2 \vee \overline{x_2})$, this is p. Labels indicate the encoding of clauses, and dotted boxes indicate $\mathsf{choice}(1)$, $\mathsf{choice}(2)$, and $\mathsf{staircase}$-components. The implementation that generated this example is accessible online [7].

3 Correctness

From Definition 1, it follows that

$$\text{START-HIROIMONO} \subsetneq \begin{matrix} \text{HIROIMONO} \\ \text{180-START-HIROIMONO} \end{matrix} \subsetneq \text{180-HIROIMONO}.$$

Thus, to prove that the map $\phi \mapsto \mathsf{p}$ from the previous section is indeed a correct reduction from 3-SAT to each of the four problems above, it suffices to show that $\phi \in$ 3-SAT $\Rightarrow \mathsf{p} \in$ START-HIROIMONO and $\mathsf{p} \in$ 180-HIROIMONO $\Rightarrow \phi \in$ 3-SAT.

3.1 Satisfiability Implies Solvability

Suppose that ϕ has a satisfying truth assignment t^*. We will solve p in two stages. First, we start at the leftmost stone ⊙ and go to the upper rightmost stone along the path $\mathsf{R}(t^*)$, where we for any truth assignment t, define $\mathsf{R}(t)$ as shown in Fig. 6–8.

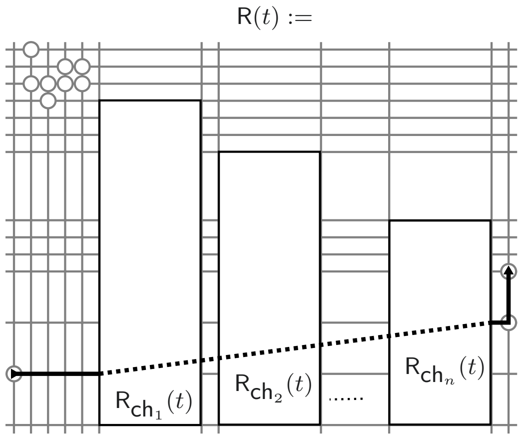

Fig. 6. The path $\mathsf{R}(t)$, which, if t satisfies ϕ, is the first stage of a solution to p

Definition 2. *Two stones on the same grid line are called* neighbors*.*

By the construction of p and R, we have the following:

Lemma 1. *For any t and k, after $\mathsf{R}(t)$, there is a stone in a $\mathsf{c}(k, 1)$-component with a neighbor in a* staircase*-component if and only if t satisfies C_k.*

In the second stage, we go back through the choice-components as shown in Fig. 9 and 10. We climb each remaining staircase by performing $\mathsf{R_{SC}}$ backwards, but whenever possible, we use the first matching alternative in Fig. 11 to "collect a clause". By Lemma 1, we can collect all clauses. See Fig. 12 for an example.

Since this two-stage solution starts from the first stone ⊙ and does not make 180° turns, it witnesses that $\mathsf{p} \in$ START-HIROIMONO.

3.2 Solvability Implies Satisfiability

Suppose that $\mathsf{p} \in$ 180-HIROIMONO, and let s be any solution witnessing this (assuming neither that s starts at the leftmost stone nor that it avoids 180° turns).

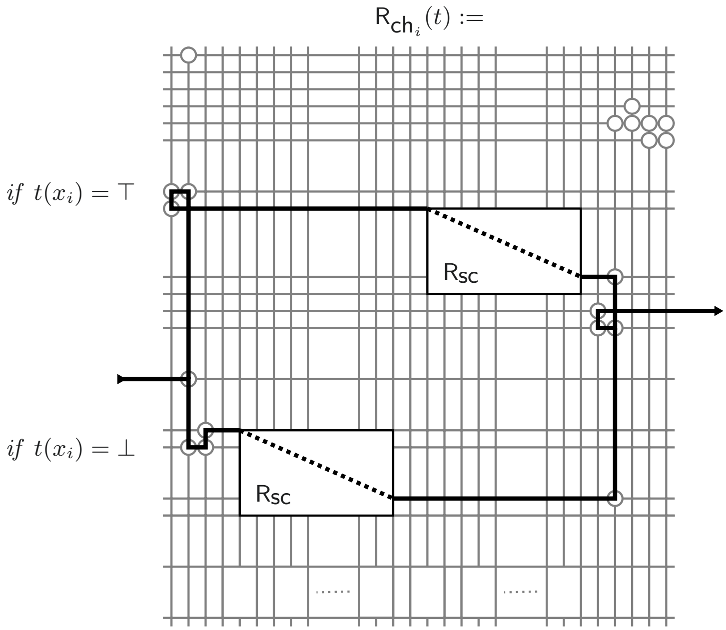

Fig. 7. Assigning a truth value by choosing the upper or lower staircase

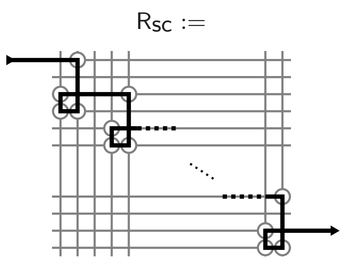

Fig. 8. Descending a staircase

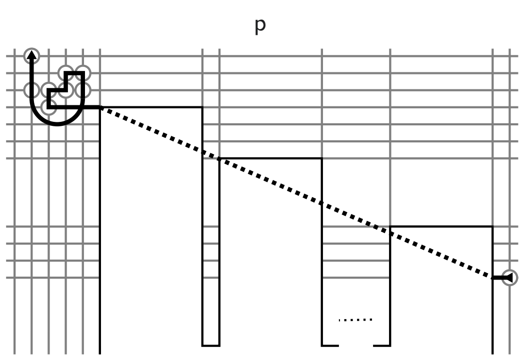

Fig. 9. The second stage of solving p

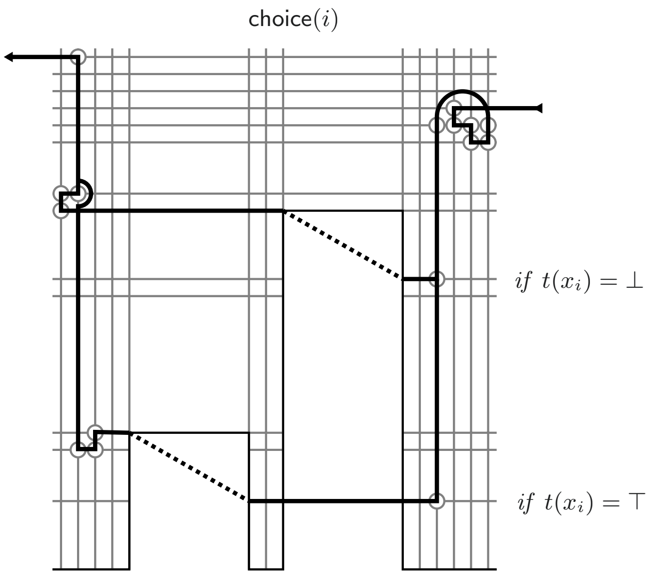

Fig. 10. In the second stage, the remaining staircase-component in choice(i) is collected

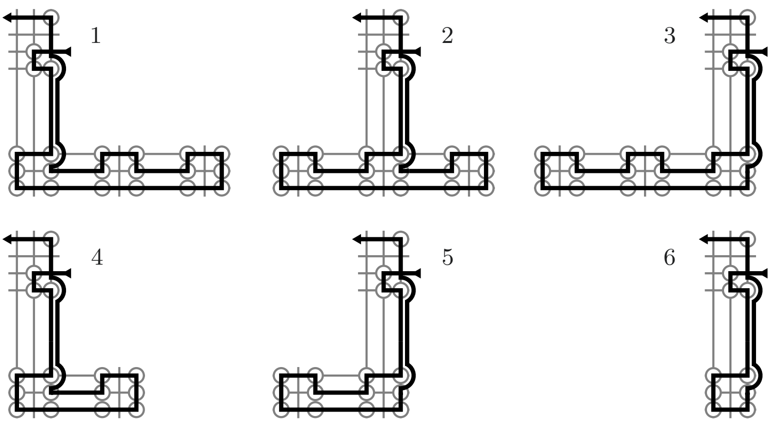

Fig. 11. Six different ways to "collect a clause" when climbing a step in a staircase

Now consider what happens as we solve p using s. Note that since the topmost stone and the leftmost one each have only one neighbor, s must start at one of these and end at the other. We will generalize this type of reasoning to *sets* of stones.

Definition 3. *A* situation *is a set of remaining stones and a current position. A* dead end D *is a nonempty subset of the remaining stones such that:*

– *There is at most one remaining stone outside of* D *that has a neighbor in* D*.*
– *No stone in* D *is on the same grid line as the current position.*

A hopeless *situation is one with two disjoint dead ends.*

Since the stones in a dead end must be the very last ones picked up, a solution can never create a hopeless situation. If we start at the topmost stone, then we

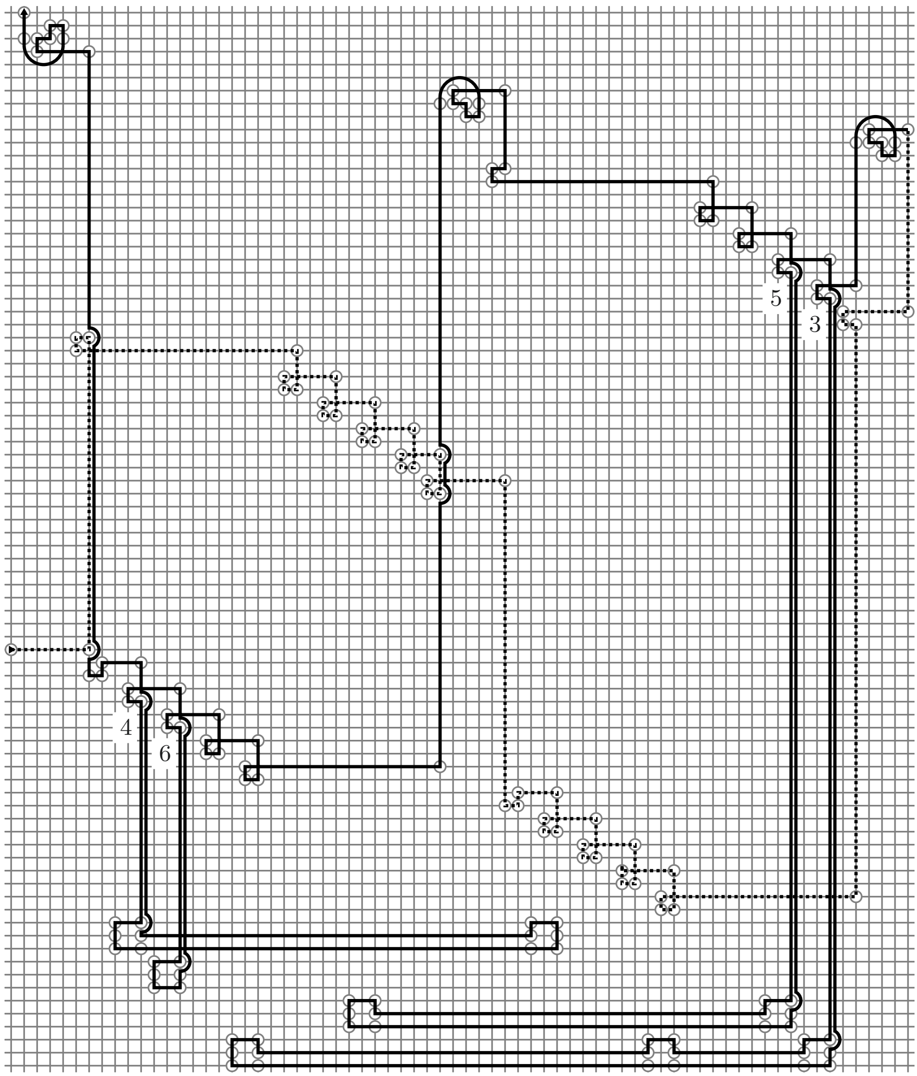

Fig. 12. A solution to the example in Fig. 5. The dotted path shows the first stage $\mathsf{R}(t^*)$, with $t^*(x_1) = \top$ and $t^*(x_2) = \bot$. The solid path shows the second stage, with numbers indicating the alternative in Fig. 11 used to collect each clause.

will after collecting at most four stones find ourselves in a hopeless situation, as is illustrated in Fig. 13. Therefore, s must start at the leftmost stone and end at the topmost one.

We claim that there is an assignment t^* such that s starts with $\mathsf{R}(t^*)$. Figure 14 shows all the ways that one might attempt to deviate from the set of R-paths and the dead ends that would arise. By Lemma 1, we have that if this t^* were to fail to satisfy some clause C_k, then after $\mathsf{R}(t^*)$, the stones in the $\mathsf{c}(k, 1)$-components would together form a dead end. We conclude that the assignment t^* satisfies ϕ.

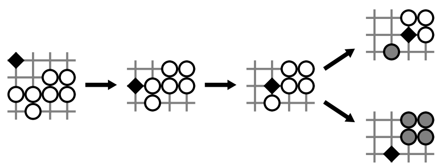

Fig. 13. Starting at the topmost stone inevitably leads to a hopeless situation. A ◆ denotes the current position, and a ● denotes a stone in a dead end.

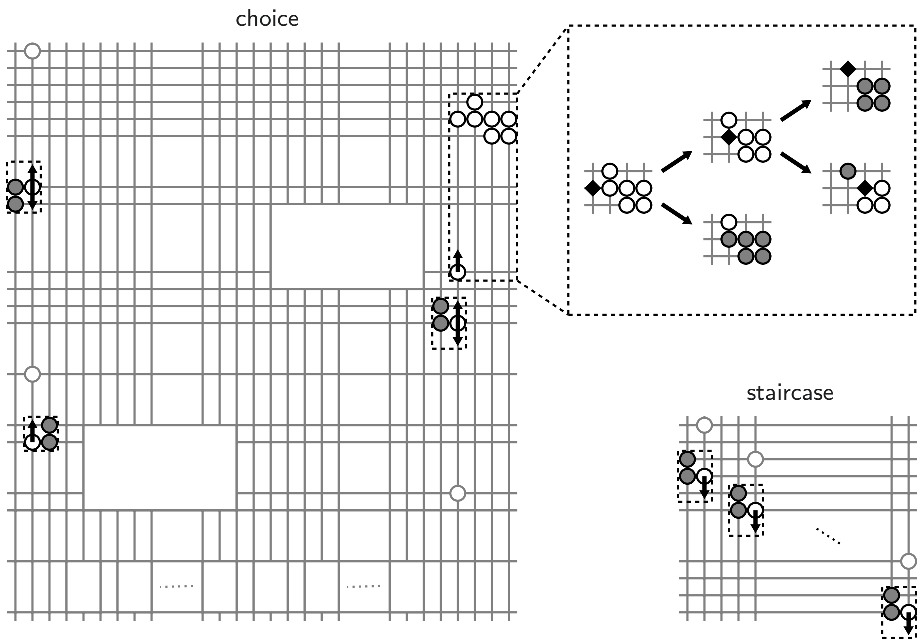

Fig. 14. Possible deviations from the R-paths and the resulting dead ends

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